Mass Transfer in Laminar-Boundary-Layer Flows with Finite Interfacial Velocities

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In the September, 1960, issue of the *A.I.Ch.E. Journal,* Andreas Acrivos gave a new method for predicting mass transfer rates toward solid surfaces (*1*). Some limitations of the method are given here, an alternate method is described, and some pertinent analogies are pointed out.

NUMERICAL COMPARISONS

For forced-convection mass transfer on arbitrary two-dimensional surfaces, Acrivos has proposed the formula

$$N_{Nu} = [(N_{Nu})_0^{3/2} + (n_{AB}(N_{Nu})_1)^{3/2}]^{2/3} \tag{1}$$

This empirical formula predicts the local diffusional Nusselt number as a function of the quantity $n_{AB} = (W_\infty - W_s)/(1 - W_s)$. The Nusselt number $(N_{Nu})_0$ for $n_{AB} \rightarrow 0$ is presumed known, and the quantity $(N_{Nu})_1$ is given by

$$(N_{Nu})_1 = \frac{U_i}{\sqrt{\int_0^{x_1} U_i\, dx_1}} \cdot \sqrt{\frac{N_{Re} N_{Sc}}{2(1-n_{AB})(1+N_{Sc})}} \tag{2}$$

Acrivos has shown that the function $(N_{Nu})_1$ reduces asymptotically to the exact solution for N_{Nu} in the limit as $n_{AB} \rightarrow 1$, for all two-dimensional flow geometries. These equations are for binary mixtures with positive n_{AB} and one stagnant (nontransferred) component.

Equation (2) is a surprisingly simple and general result. Its generality hinges on the fact that boundary-layer separation can be prevented, on surfaces of any shape, by very rapid mass transfer toward the wall (*9*). Equation (1) on the other hand is empirical, and its limitations have not been fully stated.

In Table 1, Equation (1) is compared with the exact calculations of Feyk and the writer (*3, 4, 5, 8, 10*) for steady laminar flow along a flat plate. Clearly Equation (1) holds within ± 5% for N_{Sc} near unity, but it becomes progressively less accurate at higher values of N_{Sc}. Hence Equation (1) should not be used for liquids.

The inaccuracy of Equation (1) at large values of N_{Sc} can be traced in part to Equation (2). The latter equation was derived on the assumption of a large interfacial velocity; hence it does not provide a useful asymptote for estimating N_{Nu} at large values of N_{Sc}, where the interfacial velocity is normally small.

A better method of computing N_{Nu} at high values of N_{Sc} is to use the flat-plate solutions for $N_{Nu}/(N_{Nu})_0$ from Table 1. More complete tabulations and plots for this purpose are available elsewhere (*3, 4, 8, 10*). In recent correspondence with the writer, Acrivos (*2*) has shown that this procedure is exact in the limit as $N_{Sc} \rightarrow \infty$ for steady state laminar boundary layers on any fixed two-dimensional surface, upstream of the separation point. More recently, the writer has obtained a corresponding proof for three-dimensional flow (*11*).

In view of the comparisons and new developments discussed above, it appears that a further study of geometrical effects in mass transfer problems is needed. It is hoped that a more general method of evaluating N_{Nu} can be found, which will be consistent with the exact solutions in Equation (2) and Table 1.

ANALOGIES AT HIGH MASS TRANSFER RATES

In this discussion, for the sake of clarity, some special terminology will be adopted. Mass transfer means any flow of material through an interface; this can occur either in a pure fluid (as in condensation of a pure vapor) or in a mixture. Diffusional transfer means the motion of the components of a mixture through an interface at unequal velocities. The merits of this terminology will become clearer as the discussion proceeds.

The fact that analogies can exist between heat transfer and diffusional transfer at high interfacial velocities (high mass transfer rates) is not mentioned by Dr. Acrivos. This fact is important in the application of his results and in relating them to the quantitative analogies for high mass transfer rates that are already known (*3, 4, 5, 6, 7, 8, 10*). Therefore it is useful to note that Equations (1) through (6) of Acrivos' paper describe an analogous pure-fluid heat transfer problem (for example condensation or transpiration cooling) when the following notational changes are made:

Notations in diffusion problem (Reference 1)	Notations for analogous heat transfer problem	
$\Theta = \dfrac{W - W_s}{W_\infty - W_s}$	$\Pi_T = \dfrac{T - T_s}{T_\infty - T_s}$	(3, 4)
$N_{Sc} = \dfrac{\mu}{\rho \mathcal{D}_{AB}} = \Lambda_{AB}$	$N_{Pr} = \dfrac{\hat{C}_p \mu}{k} = \Lambda_T$	(5, 6)
$n_{AB} = -\dfrac{v_1 N_{Sc}}{(\partial \Theta/\partial y_1)}\Bigg\vert_{y_1=0} = \dfrac{W_\infty - W_s}{1 - W_s}$	$n_T = -\dfrac{v_1 N_{Pr}}{(\partial \Pi_T/\partial y_1)}\Bigg\vert_{y_1=0} = \dfrac{v_0 \rho \hat{C}_p (T_\infty - T_s)}{q_0}$	(7, 8)

Here it will be noted that the heat transfer problem includes mass transfer of the pure fluid through the interface, since Equation (8) contains the interfacial velocity v_0 (or $v_1|_{y_1=0}$ in dimensionless notation). There is of course no diffusion in the heat transfer problem since a pure fluid is being considered.

When the above substitutions are made, one finds that the x-momentum equation is unchanged:

$$u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} = U_i \frac{dU_i}{dx_i} + \frac{\partial^2 u_1}{\partial y_1^2} \tag{9}$$

The mass balance is unchanged:

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} = 0 \tag{10}$$

The diffusion equation is replaced by the energy equation (with viscous dissipation neglected):

$$u_1 \frac{\partial \Pi_T}{\partial x_1} + v_1 \frac{\partial \Pi_T}{\partial y_1} = \frac{1}{N_{Pr}} \frac{\partial^2 \Pi_T}{\partial y_1^2} \tag{11}$$

Table 1. Comparison of Predicted Values of $N_{Nu}/(N_{Nu})_0$ for Laminar Flow Along a Flat Plate

	At $N_{Sc} = 0.6$			At $N_{Sc} = 2.0$			At $N_{Sc} = 100$			At $N_{Sc} = \infty$		
n	Exact solution	Equation (1)	Error, %	Exact solution	Equation (1)	Error, %	Exact solution	Equation (1)	Error, %	Exact solution	Equation (1)	Error, %
0	1.00	1.00	—	1.00	1.00	—	1.00	1.00	—	1.00	1.00	—
0.3	1.30	1.26	−3.1	1.29	1.22	−5.5	1.22	1.04	−15	1.22	1.00	−18
0.6	1.93	1.99	3.2	1.82	1.83	0.5	1.63	1.18	−28	1.63	1.00	−61
0.8	3.01	3.18	5.5	2.75	2.86	3.8	2.3	1.43	−38	2.27	1.00	−56
0.9	4.55	4.76	4.6	4.05	4.22	4.3	3.1	1.81	−42	3.08	1.00	−67

The boundary conditions become

$$\text{At } y_1 = 0: \quad u_1 = 0 \quad \Pi_T = 0 \quad (12a)$$

$$\text{At } y_1 = \infty: u_1 = U_i(x_1) \quad \Pi_T = 1 \quad (12b)$$

$$\text{At } x_1 = 0: \quad u_1 = U_i(0) \quad \Pi_T = 1 \quad (12c)$$

$$\text{At } y_1 = 0: \quad v_1 = -\frac{n_T}{N_{Pr}}\frac{\partial \Pi_T}{\partial y_1} \quad (12d)$$

Here the dependence of ρ, μ, $\hat{C}_p$, and k on pressure and temperature is neglected.

From the complete analogy of Equations (9) through (12*d*) with Acrivos' equations (1) through (4*d*) it follows that the solutions for Θ and Π_T are analogous. His Equation (6) may then be written in the following form, valid for pure-fluid heat transfer or isothermal diffusional transfer in steady laminar flow:

$$\frac{N_{Nu}}{\sqrt{N_{Re}}} = F(\Lambda, x_1, n) \quad (13)$$

in which N_{Nu}, Λ, and n are to be evaluated for the governing transport process. Thus one sees that an exact analogy does exist between forced-convection heat transfer and diffusional transfer, even at high mass transfer rates.

Equation (13) includes, as a special case, the conditions for which Equation (2) was derived; hence Equation (2) may be written in the more general form:

$$\lim_{n \to 1} N_{Nu}\sqrt{1-n} = \frac{U_i}{\sqrt{\int_0^{x_1} U_i\, dx_1}} \sqrt{\frac{N_{Re}\,\Lambda}{2(1+\Lambda)}} \quad (14)$$

in which the factor $\sqrt{1-n}$ has been transposed to give a finite limit. A similar generalization applies to Acrivos' Equation (36); one simply replaces N_{Sc} by Λ and n_{AB} by n and interprets N_{Gr} as a thermal or diffusional Grashof number. Again the result is valid for pure-fluid heat transfer or isothermal diffusional transfer.

Several classes of problems exist for which the assumed boundary conditions on u_1, v_1, Π_T, and Θ can be superimposed. In such cases, with the added restrictions that $\hat{C}_{pA} = \hat{C}_{pB}$ and that there is no heat of mixing, Equation (13) applies exactly to simultaneous heat and diffusional transfer. Equation (14) can be so applied in the limit as both n_T and n_{AB} approach unity. Solutions for wider ranges of n_T and n_{AB} are available for flat-plate flow and stagnation flow (*3, 4, 5, 6, 7, 8, 10*), and an extensive tabulation for other wedge flows has recently been completed (*12*). These solutions can all be analogously applied to the diffusional transfer of both components in a binary system, with simultaneous heat transfer (*3, 4, 8, 10, 12*).

Further information on interfacial-velocity effects is given in Reference *4*. There the exact results displayed in Table 1 are more fully tabulated for both positive and negative *n*, so that mass transfer into or out of the stream can be treated. The analogous problems of diffusional transfer of both components in a binary system, with simultaneous heat transfer, are included by appropriate generalizations of the quantities n_T and n_{AB}. Estimation procedures for turbulent flow and multicomponent systems are presented. These results provide a simple and systematic approach to a variety of mass transfer problems.

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NOTATION

- $\hat{C}_p$ = heat capacity of the fluid per unit mass
- $\mathcal{D}_{AB}$ = diffusion coefficient in a binary fluid
- k = thermal conductivity of the fluid
- L = characteristic length of the surface
- q_0 = heat flux in y direction at the interface
- T = temperature
- u = x component of the local fluid velocity
- u_1 = u/U_∞
- U_∞ = characteristic velocity of the fluid
- $U_i(x_1)$ = potential flow velocity distribution divided by U_∞
- v = y component of the local fluid velocity
- v_1 = $(v/U_\infty)\sqrt{N_{Re}}$
- v_0 = velocity of the fluid through the interface
- x = distance along the interface from the leading edge
- x_1 = x/L
- y = distance from the interface into the fluid
- y_1 = $(y/L)\sqrt{N_{Re}}$
- W = weight fraction of the transferred species
- W_s, W_∞ = weight fractions on the stream side of the interface and at $y_1 = \infty$, respectively
- N_{Nu} = local Nusselt number based on characteristic length L; for diffusion $N_{Nu} = L(\partial\Theta/\partial y)|_{y=0}$ and for heat transfer $N_{Nu} = L(\partial\Pi_T/\partial y)|_{y=0}$
- N_{Pr} = Prandtl number, $\hat{C}_p\mu/k$
- N_{Re} = Reynolds number, $U_\infty L\rho/\mu$
- N_{Sc} = Schmidt number, $\mu/\rho\mathcal{D}_{AB}$

Greek Letters

- Θ = $(W - W_s)/(W_\infty - W_s)$
- Λ = N_{Pr} for heat transfer or N_{Sc} for diffusion
- Π_T = $(T - T_s)/(T_\infty - T_s)$
- ρ = fluid density
- μ = fluid viscosity

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